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ADDENDUM

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Abstract. The formalism of the generalized fermionic Bogolyubov transformation, previously by Fan and VanderLinde, is extended to the bosonic case. The corresponding bosonic quasi-particle and quasi-vacuum state are derived; the bosonic generalized Bogolyubov operator is decomposed as a normal product form by the technique of integration within an ordered product (IWOP).

1. Introduction

In a previous work [1], by exploiting the newly developed technique of integration within ordered products (IWOP) [2] we extended the fermionic Bogolyubov transformation (frequently used in dealing with pairing interactions of fermions) to allow for binary coupling between any (even) number of fermions by using a matrix transformation 'coefficient'. In this addendum we generalize the formalism of [1] to the boson case. The boson Bogolyubov transformation has been used in quantum statistics [3] and the quantum theory of magnetism [4]; however, the corresponding unitary operator which can engender multimode Bogolyubov transformation has not yet received enough attention in the literature. Similar in spirit to the derivation in [1], in the following we shall derive the normally ordered expansion of the bosonic generalized Bogolyubov operator $U(G)=\exp\{\frac{1}{2}(a_iG_{ij}a_j-a_i^\dagger G_{ij}^\dagger a_j)\}$ $(i,j=1,2,\ldots,n)$, where we have adopted the Einstein convention (if an index is repeated in a term, summation over it from 1 to n is implied) and G is a complex symmetric matrix of course. In section 2 we show that the operator $U(G)$ generates the Bogolyubov transformation whose 'coefficient' is a matrix. In section 3, the boson quasi-particle vacuum state is deduced. In section 4, with the help of the IWOP technique and using the coherent state representation [5] we derive the normal product form of $U(G)$.

2. Transformation properties of boson creator $a_i^\dagger$ under $U(G)$

Using the commutator result $[a_i, a_j^\dagger]=\delta_{ij}$, and the operator identity

$$e^A B\, e^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\frac{1}{3!}[A,[A,[A,B]]]+\ldots \tag{1}$$

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we obtain

$$Ua_i^\dagger U^{-1} = a_j^\dagger[\cosh(G^\dagger G)^{1/2}]_{ji} + a_j[G(G^\dagger G)^{-1/2}\sinh(G^\dagger G)^{1/2}]_{ji} \qquad (2.1)$$

$$= a_j^\dagger[\cosh(G^\dagger G)^{1/2}]_{ji} + a_j[\sinh(GG^\dagger)^{1/2}(GG^\dagger)^{-1/2}G]_{ji}. \qquad (2.2)$$

As shown in [6], an arbitrary non-singular matrix can always be decomposed as the product of a unitary matrix and a Hermitian matrix. Then in a fashion analogous to the polar representation of complex numbers, we may in general write

$$G = H\,\mathrm{e}^{\mathrm{i}F} \qquad (3)$$

with $H = H^\dagger$, $F = F^\dagger$. Since $\tilde{G} = G$, we also have

$$\begin{aligned} &G = \mathrm{e}^{\mathrm{i}\tilde{F}}\tilde{H} \qquad \widetilde{G^\dagger G} = GG^\dagger \qquad \widetilde{GG^\dagger G} = GG^\dagger G \\ &GG^\dagger = H^2 \qquad G^\dagger G = \tilde{H}^2 \qquad \tilde{H}^2\,\mathrm{e}^{-\mathrm{i}F} = \mathrm{e}^{-\mathrm{i}F}H^2. \end{aligned} \qquad (4)$$

Equation (2) can then be put into the form:

$$Ua_i^\dagger U^{-1} = a_j^\dagger(\cosh\tilde{H})_{ji} + a_j(\mathrm{e}^{\mathrm{i}\tilde{F}}\sinh\tilde{H})_{ji} \qquad (5.1)$$

$$= a_j^\dagger(\cosh\tilde{H})_{ji} + a_j[(\sinh H)\,\mathrm{e}^{\mathrm{i}F}]_{ji} \equiv a_i'^\dagger. \qquad (5.2)$$

It then follows that

$$\begin{aligned} Ua_iU^{-1} &= a_j(\cosh H)_{ji} + a_j^\dagger(\mathrm{e}^{-\mathrm{i}F}\sinh H)_{ji} \\ &= a_j(\cosh H)_{ji} + a_j^\dagger(\sinh\tilde{H}\,\mathrm{e}^{-\mathrm{i}\tilde{F}})_{ji} = a_i'. \end{aligned} \qquad (6)$$

Note that $(\mathrm{e}^{-\mathrm{i}F}\sinh H)$ is a symmetric matrix.

3. Multimode boson quasi-particle vacuum state generated by U

We now seek the boson quasi-particle vacuum state $U|\vec{0}\rangle \equiv \|\vec{0}\rangle$ annihilated by a_l', where $|\vec{0}\rangle$ is the multi-boson vacuum state annihilated by a_l, $a_l|\vec{0}\rangle = 0$. For this purpose, we first establish an equation satisfied by $\|0\rangle$ by allowing a_l to operate on $\|\vec{0}\rangle$

$$a_l\|\vec{0}\rangle = UU^{-1}a_lU|\vec{0}\rangle \qquad (7)$$

which we then solve to get $\|\vec{0}\rangle$. As a result of (5) and (6) as well as $U^\dagger(G) = U(-G)$, we may express (7) as

$$\begin{aligned} a_l\|\vec{0}\rangle &= U[a_j(\cosh H)_{jl} - a_j^\dagger(\mathrm{e}^{-\mathrm{i}F}\sinh H)_{jl}]|\vec{0}\rangle \\ &= -Ua_j^\dagger(\mathrm{e}^{-\mathrm{i}F}\sinh H)_{jl}U^{-1}U|\vec{0}\rangle \\ &= -[a_i^\dagger(\cosh\tilde{H}\,\mathrm{e}^{-\mathrm{i}F}\sinh H)_{il} + a_i(\sinh^2 H)_{il}]\|\vec{0}\rangle. \end{aligned} \qquad (8)$$

It then follows from (4) and (8) that

$$a_l\|\vec{0}\rangle = -a_j^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{jl}\|\vec{0}\rangle \qquad (9)$$

which is the equation we need for $\|\vec{0}\rangle$ to obey. By noticing

$$\begin{aligned} &[a_i, \exp(-\tfrac{1}{2}a_i^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{ij}a_j^\dagger)] \\ &\quad = -a_j^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{ji}\exp(-\tfrac{1}{2}a_i^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{ij}a_j^\dagger) \end{aligned} \qquad (10)$$

we solve equation (9) to obtain

$$\|\vec{0}\rangle = C\exp[-\tfrac{1}{2}a_i^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{ij}a_j^\dagger]|\vec{0}\rangle. \qquad (11)$$

The normalization factor C can be derived by evaluating the norm

$$1=\langle\vec{0}\|\vec{0}\rangle=|C|^2\langle\vec{0}|\exp[-\tfrac{1}{2}a_i(\tanh H\,e^{iF})_{ij}a_j]\exp[-\tfrac{1}{2}a_i^\dagger(e^{-iF}\tanh H)_{ij}a_j^\dagger]|\vec{0}\rangle. \tag{12}$$

In order to put the anti-normally ordered exponential operators in (12) into normal product form so that the vacuum state expectation value can be readily obtained, we use the operator identity deduced in [7]

$$\begin{aligned}e^{a_i\sigma_{ij}a_j}\,e^{a_i^\dagger\tau_{ij}a_j^\dagger}&=[\det(\mathbb{1}-4\sigma\tau)]^{-1/2}\exp\{a_i^\dagger[(\mathbb{1}-4\tau\sigma)^{-1}\tau]_{ij}a_j^\dagger\}\\&\quad\times{:}\exp[a_i^\dagger(\mathbb{1}-4\tau\sigma)^{-1}_{ij}a_j-a_i^\dagger a_i]{:}\\&\quad\times\exp\{a_i[(\mathbb{1}-4\sigma\tau)^{-1}\sigma]_{ij}a_j\}\end{aligned} \tag{13}$$

where σ and τ are both $n\times n$ symmetric matrices, and $\mathbb{1}$ is the $n\times n$ unit matrix. As a result of employing (13), from (12) we obtain, up to a phase factor,

$$C=[\det(\operatorname{sech} H)]^{1/2}. \tag{14}$$

4. Normally ordered expansion of $U(G)$

We now seek the normal product form of $U(G)$. By introducing the overcompleteness relation of the boson coherent state

$$\int\frac{d^2\vec{z}}{\pi}|\vec{z}\rangle\langle\vec{z}|=\int\prod_i\left[\frac{d^2z_i}{\pi}\right]{:}\exp[-(z_i^*-a_i^\dagger)(z_i-a_i)]{:}=1 \tag{15}$$

where $|\vec{z}\rangle$ is defined as

$$|\vec{z}\rangle\equiv|z_1,z_2,\ldots,z_n\rangle=\exp[-\tfrac{1}{2}z_i^*z_i+z_ia_i^\dagger]|\vec{0}\rangle\equiv\exp[-\tfrac{1}{2}|\vec{z}|^2+\vec{z}\vec{a}^\dagger]|\vec{0}\rangle \tag{16}$$

we can rewrite $U(G)$ as

$$\begin{aligned}U|\vec{z}\rangle&=U\exp(z_ia_i^\dagger)U^{-1}U|\vec{0}\rangle\exp(-\tfrac{1}{2}|z_i|^2)\\&=[\det(\operatorname{sech} H)]^{1/2}\exp\{z_i[a_j^\dagger(\cosh\tilde{H})_{ji}+a_j(\sinh H\,e^{iF})_{ji}]\}\\&\quad\times\exp\{-\tfrac{1}{2}a_i^\dagger(e^{-iF}\tanh H)_{ij}a_j^\dagger-\tfrac{1}{2}|z_i|^2\}|\vec{0}\rangle.\end{aligned} \tag{17}$$

In deriving (17) we have used (5) and (11). Using the Baker–Hausdorff formula and (10) we can decompose the first exponential in (17) as

$$\exp\{\ldots\}=\exp(z_i(\cosh H)_{ij}a_j^\dagger)\exp(a_j(\sinh H\,e^{iF})_{ji}z_i)\exp(\tfrac{1}{4}z_i(\sinh 2H\,e^{iF})_{ij}z_j). \tag{18}$$

Therefore, (17) becomes

$$\begin{aligned}U|\vec{z}\rangle=[\det(\operatorname{sech} H))^{1/2}\exp[&-\tfrac{1}{2}|z_i|^2+z_i(\operatorname{sech} H)_{ij}a_j^\dagger+\tfrac{1}{2}z_i(\tanh e^{iF})z_j\\&-\tfrac{1}{2}a_i^\dagger(e^{-iF}\tanh H)_{ij}a_j^\dagger]|\vec{0}\rangle.\end{aligned} \tag{19}$$

By virtue of (19) and the IWOP technique we can further express $U(G)$ as

$$\begin{aligned}U&=\int\frac{d^2\vec{z}}{\pi}U|\vec{z}\rangle\langle\vec{z}|\\&=[\det(\operatorname{sech} H)]^{1/2}\int\prod_i\left[\frac{d^2z_i}{\pi}\right]\\&\quad\times{:}\exp\{-|z_i|^2+z_i(\operatorname{sech} H)_{ij}a_j^\dagger+z_i^*a_i+\tfrac{1}{2}z_i(\tanh H\,e^{iF})_{ij}z_j\\&\quad-\tfrac{1}{2}a_i^\dagger(e^{-iF}\tanh H)_{ij}a_j^\dagger-a_i^\dagger a_i\}{:}.\end{aligned} \tag{20}$$

In terms of the integration formula [7, 8]

$$\int \prod_{i=1}^{n}\left[\frac{d^2 z_i}{\pi}\right]\exp\left[-\tfrac{1}{2}(\vec{z},\vec{z}^*)\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\vec{z}\\ \vec{z}^*\end{pmatrix}+(\vec{\mu},\vec{\nu}^*)\begin{pmatrix}\vec{z}\\ \vec{z}^*\end{pmatrix}\right]$$
$$=\left[\det\begin{pmatrix}C & D\\ A & B\end{pmatrix}\right]^{-1/2}\exp\left[\tfrac{1}{2}(\vec{\mu},\vec{\nu}^*)\begin{pmatrix}C & D\\ A & B\end{pmatrix}^{-1}\begin{pmatrix}\vec{\nu}^*\\ \mu\end{pmatrix}\right] \tag{21}$$

where A, B, C, D are all square matrices of order n, $\tilde{B}=B$, $\tilde{C}=C$,, and the method of parttitioning of matrices for finding the inverse and the determinant of a matrix

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}^{-1}=\begin{pmatrix}(A-BD^{-1}C)^{-1} & A^{-1}B(CA^{-1}B-D)^{-1}\\ D^{-1}C(BD^{-1}C-A)^{-1} & (D-CA^{-1}B)^{-1}\end{pmatrix}$$
$$\det\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\det A\,\det(D-CA^{-1}B) \tag{22}$$

then equation (20) finally becomes

$$U=[\det(\operatorname{sech} H)]^{1/2}\int\prod_i\left[\frac{d^2 z_i}{\pi}\right]:\exp\left\{-\tfrac{1}{2}(\vec{z},\vec{z}^*)\begin{pmatrix}-\tanh H\,e^{iF} & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}\vec{z}\\ \vec{z}^*\end{pmatrix}\right.$$
$$\left.+(\vec{a}^\dagger\operatorname{sech}\tilde{H},\vec{a})\begin{pmatrix}\vec{z}\\ \vec{z}^*\end{pmatrix}-a_i^\dagger a_i-\tfrac{1}{2}a_i^\dagger(e^{-iF}\tanh H)_{ij}a_j^\dagger\right\}:$$
$$=[\det(\operatorname{sech} H)]^{1/2}\exp[-\tfrac{1}{2}a_i^\dagger(e^{-iF}\tanh H)_{ij}a_j^\dagger]$$
$$\times:\exp[a_i^\dagger(\operatorname{sech}\tilde{H}-\mathbb{1})_{ij}a_j]:\exp[\tfrac{1}{2}a_i(\tanh H\,e^{iF})_{ij}a_j]. \tag{23}$$

By virtue of the formula

$$:\exp[a_i^\dagger(e^{\Lambda}-\mathbb{1})_{ij}a_j]:a_l:\exp[a_i^\dagger(e^{-\Lambda}-\mathbb{1})_{ij}a_j]:=(e^{-\Lambda})_{li}a_i \tag{24}$$

and $\tilde{H}^2 e^{-iF}=e^{-iF}H^2$, one can check that the result (23) indeed generates the transformations (5) and (6). As a direct application of (23), following Bogolyubov [3] and Tyablikov [4] we consider the Hamiltonian [9]

$$\mathcal{H}=a_i^\dagger L_{ij}a_j+\tfrac{1}{2}(a_i^\dagger M_{ij}a_j^\dagger+a_i M_{ij}^\dagger a_j)$$

where $L=L^\dagger$, $M=\tilde{M}$. In terms of (5) and (6) one can diagonalize $\mathcal{H}$ as $\mathcal{H}=E^{(0)}+E^{(i)}a_i'^\dagger a_i'$, where the energies $E^{(i)}$ and the matrices H and F are determined by

$$(E^{(i)}\delta_{lj}-L_{lj}^*)(\cosh\tilde{H})_{ij}=-(\sinh\tilde{H}\,e^{-i\tilde{F}})_{ij}M_{jl}^*$$
$$(E^{(i)}\delta_{lj}+L_{lj})(\sinh\tilde{H}\,e^{-i\tilde{F}})_{ij}=(\cosh\tilde{H})_{ij}M_{jl}$$

which can lead us to find the energy E^0

$$E^{(0)}=-E^{(i)}(\sinh^2 H)_{ii}.$$

In summary, by combining and contrasting this work and the formalism of [1], we have developed the original Bogolyubov transformation to a new formalism in which the parameter is a matrix. The IWOP techniques for both boson and fermion systems have played an essential role in both [1] and the present addendum.

Note added in proof. Using IWOP, we can also directly perform the integration

$$\int \mathrm{d}^n\vec{q}|\mathrm{e}^{\Lambda}\vec{q}\rangle\langle q^2|(\det \mathrm{e}^{\Lambda})^{1/2}$$

$$= [\det \operatorname{sech} \Lambda]^{1/2} \exp[\tfrac{1}{2}a_i^\dagger(\tanh \Lambda)_{ij}a_j^\dagger]$$

$$\times :\exp[a_i^\dagger(\operatorname{sech} \Lambda - \mathbb{1})_{ij}a_j]: \exp[-\tfrac{1}{2}a_i(\tanh \Lambda)_{ij}a_j]$$

where Λ is a real symmetric matrix, $\mathrm{e}^{2\Lambda} + \mathbb{1}$ is positive definite, and $|\vec{q}\rangle$ is the n-mode coordinate eigenstate. This is a special case of (23).

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